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2004 J. Phys. A: Math. Gen. 37 5729

(http://iopscience.iop.org/0305-4470/37/22/002)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 5729–5736 PII: S0305-4470(04)76259-9

Underdamped oscillator with fluctuating damping

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Received 16 February 2004
Published 18 May 2004
Online at stacks.iop.org/JPhysA/37/5729
DOI: 10.1088/0305-4470/37/22/002

Abstract

The second moments of a harmonic oscillator with a periodically and randomly varying damping parameter and periodic driving force are found for white and dichotomous noise.

PACS numbers: 02.50.Ey, 05.40.−a, 05.45.−a

1. Introduction

The studies of noise-induced dynamics are high on the list of outstanding problems in statistical mechanics. Such phenomena as stochastic resonance have found many applications in physics and biology [1]. Since nonlinearity presents difficulties for theoretical analysis, linear models with multiplicative noise are of special interest. These models, on the one hand, show quasi-nonlinear behaviour including stochastic resonance [2, 3], and on the other hand, allow an exact analytical treatment. The linearity of the problems permits the inclusion of inertial effects which are usually neglected. In this way, we are led to the problem of a periodically driven underdamped harmonic oscillator with multiplicative noise. A model that is much studied [4] is that of a harmonic oscillator with a random frequency

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega^2[1 + \xi(t)]x = a\sin(\Omega t) + f(t) \tag{1}$$

where $\xi(t)$ and $f(t)$ represent the multiplicative and additive noise, respectively.

Although this model has a long history [5], further research on both the internal dynamics [6] and the response to the driving force [7] is still in progress.

There is an increasing number of problems where the particles advected by the mean flow pass through the region under study. These include problems of phase transitions under shear [8], open flows of liquids [9], Rayleigh–Benard and Taylor–Couette problems in fluid dynamics [10], dendritic growths [11], chemical waves [12] and the motion of vortices [13]. The velocity which enters the convective term is subject to fluctuations, i.e. the question arises of the isomorphic problem of a harmonic oscillator with random damping. The same problem appears when one studies the linear stability of nonlinear (say, Duffing or Van der Pol) oscillators. All the above-mentioned phenomena are described by the equation of the periodically driven underdamped harmonic oscillator with a random damping parameter,

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \gamma[1 + \xi(t)]\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = a\sin(\Omega t) + f(t) \tag{2}$$

where the random force $\xi(t)$ is a Gaussian variable with zero mean and white noise correlator

$$\langle \xi(t)\xi(t_1)\rangle = D\delta(t - t_1) \tag{3}$$

or with the exponential correlator

$$\langle \xi(t)\xi(t_1)\rangle = \sigma^2 \exp(-\lambda|t - t_1|) \tag{4}$$

which later will be assumed to be dichotomous noise. In the limit case $\sigma^2 \to \infty$ and $\lambda \to \infty$ with $\sigma^2/\lambda = D$, equation (4) transforms into (3).

We assume zero initial conditions for equation (2). Therefore, in order to get nonzero stationary solutions, we take into account the additive noise $f(t)$ for the field-free case $a = 0$, while for the driven oscillator there is no need for additive noise, and we put $f = 0$. The additive noise is taken to be white noise

$$\langle f(t) f(t_1)\rangle = D_1\delta(t - t_1)$$

with no correlation with the multiplicative noise. For thermal noise, $D_1 = \kappa T\gamma$.

However, the general properties of equation (2) have not yet been adequately investigated. In fact, we know only one article concerning white noise acting on the velocity, thereby influencing damping, and this concerned water waves influenced by a turbulent wind field [14]. The aim of this work is to make a detailed study of equation (2) which we already started in our previous work [15], where the first moment $\langle x(t)\rangle$ was found for both free motion ($a = f = 0$) and the full equation (2). In the former case, under the simplest form of the splitting of averages, it was found that in the case of white noise one obtains [15]

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \gamma(1 - \gamma D)\frac{\mathrm{d}}{\mathrm{d}t} + \omega^2\right]\langle x\rangle_0 = 0 \tag{5}$$

i.e. the presence of white noise in the original equation (2) leads to a decrease of damping if $\gamma D < 1$. Moreover, if $\gamma D > 1$, i.e. if the noise is sufficiently strong, the effective damping becomes negative, so that the average value of the coordinate x increases in time, which indicates an instability.

A similar effect exists for the case of a random frequency described by equation (1) where strong colour noise (and not white noise as in (5)) may result not only in the well-known instability of the second moments, but also in an instability of the oscillator coordinate (first moment) if the strength of the fluctuations is sufficiently large. Thus, for an undamped system this strength has to be larger than twice the unperturbed frequency [16].

Later we will need the full solution of the driven oscillator described by equation (2) with $f = 0$. This solution is the sum of the force-free solution of equation (5) and the output signal induced by an external field which has the form $\langle x\rangle_a = A\sin(\Omega t + \phi)$. For the case of white noise, the amplitude A and the phase ϕ of this signal are equal to [15]

$$A = a[(\Omega^2 - \omega^2)^2 + \gamma^2\Omega^2(1 - D\gamma)^2]^{-\frac{1}{2}} \qquad \tan\phi = \frac{\Omega\gamma(1 - D\gamma)}{\Omega^2 - \omega^2} \tag{6}$$

i.e., the amplitude has a maximum at the limit $D\gamma = 1$ of the convergence of the process $\langle x(t)\rangle$. More complicated equations equivalent to (5), (6) have also been obtained for dichotomous noise (4) [15]. To this end, we used the well-known Shapiro–Loginov procedure [17] which for exponentially correlated noise (4) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi g\rangle = \left\langle \xi \frac{\mathrm{d}g}{\mathrm{d}t}\right\rangle - \lambda\langle \xi g\rangle \tag{7}$$

where g is some function of noise, $g = g\{\xi\}$. If $\frac{\mathrm{d}g}{\mathrm{d}t} = B\xi$, then equation (7) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi g\rangle = B\langle \xi^2\rangle - \lambda\langle \xi g\rangle \tag{8}$$

and for white noise ($\xi^2 \to \infty$ and $\lambda \to \infty$ with $\xi^2/\lambda = D$), one gets

$$\langle \xi g\rangle = BD. \tag{9}$$

This paper is organized as follows: in section 2 we consider the second moments and correlation functions for equation (2) and compare them with those for equation (1). In the next two sections we consider separately the force-free oscillator (section 3) both for white and dichotomous noise, and a periodically driven oscillator (section 4). In section 5 we compare oscillators with a fluctuating and periodically exciting damping parameter. Section 6 contains a discussion of the results obtained.

2. Second moments

Equation (2) can be rewritten as two first-order differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\gamma y - \gamma\xi y - \omega^2 x + a\sin(\Omega t) + f(t). \tag{10}$$

Multiplying the first of these equations by $2x$ and the second one by $2y$, and averaging, one gets

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle x^2\rangle = 2\langle xy\rangle \\
&\frac{\mathrm{d}}{\mathrm{d}t}\langle y^2\rangle = -2\gamma\langle y^2\rangle - 2\gamma\langle \xi y^2\rangle - 2\omega^2\langle xy\rangle + 2a\langle y\rangle\sin(\Omega t) + 2D_1.
\end{aligned} \tag{11}$$

Analogously, multiplying equations (10) by y and x, respectively, summarizing and averaging the sum leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle xy\rangle = \langle y^2\rangle - \gamma\langle xy\rangle - \gamma\langle \xi xy\rangle - \omega^2\langle x^2\rangle + a\langle x\rangle\sin(\Omega t) \tag{12}$$

In deriving (11), (12), we have used equation (9) for correlators containing white noise $f(t)$, which gives $\langle yf(t)\rangle = D_1$ and $\langle xf(t)\rangle = 0$. In addition, equations (11), (12) contain new correlators $\langle \xi y^2\rangle$ and $\langle \xi xy\rangle$. One can calculate these and the analogous correlator $\langle \xi x^2\rangle$ using the Shapiro–Loginov procedure (7) for $g = x^2$, $g = xy$ and $g = y^2$, respectively. Splitting the higher-order correlators into the lower-order ones involves different approximations. We restrict our consideration to the simplest case of white and dichotomous noises where the higher-order correlators containing ξ^2 can be split by using $\xi^2 = \sigma^2$ so that, for example, $\langle \sigma^2 x^2\rangle = \langle \sigma^2\rangle\langle x^2\rangle$. Finally, one gets

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi x^2\rangle &= 2\langle \xi xy\rangle - \lambda\langle \xi x^2\rangle \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi y^2\rangle &= -2\gamma\langle \xi y^2\rangle - 2\gamma\sigma^2\langle y^2\rangle - 2\omega^2\langle \xi xy\rangle + 2a\langle \xi y\rangle\sin(\Omega t) - \lambda\langle \xi y^2\rangle \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi xy\rangle &= \langle \xi y^2\rangle - \gamma\sigma^2\langle xy\rangle - \gamma\langle \xi xy\rangle - \omega^2\langle \xi x^2\rangle + a\langle \xi x\rangle\sin(\Omega t) - \lambda\langle \xi xy\rangle.
\end{aligned} \tag{13}$$

We thus obtain a system of six equations (11)–(13) for six variables, $\langle x^2\rangle$, $\langle y^2\rangle$, $\langle xy\rangle$, $\langle \xi x^2\rangle$, $\langle \xi y^2\rangle$ and $\langle \xi xy\rangle$.

In the simple case of white noise one can use the simplified splitting procedure (8) for correlators $\langle \xi y^2\rangle$, $\langle \xi xy\rangle$, in equations (11), (12) which gives

$$\langle \xi y^2\rangle \approx -2\gamma D\langle y^2\rangle \qquad \langle \xi xy\rangle \approx -\gamma D\langle xy\rangle. \tag{14}$$

Substituting (14) into (11), (12), one obtains

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle x^2\rangle &= 2\langle xy\rangle \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle y^2\rangle &= -2\gamma\langle y^2\rangle + 4\gamma^2 D\langle y^2\rangle - 2\omega^2\langle xy\rangle + 2a\langle y\rangle\sin(\Omega t) + 2D_1 \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle xy\rangle &= \langle y^2\rangle - \gamma\langle xy\rangle + \gamma^2 D\langle xy\rangle - \omega^2\langle x^2\rangle + a\langle x\rangle\sin(\Omega t).
\end{aligned} \tag{15}$$

Note that for white noise one can find the exact expressions for higher moments of the coordinate and velocity of the form $\langle x^{m-n}y^n\rangle$. The derivative of the latter expression has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x^{m-n}y^n\rangle = (m-n)\left\langle x^{m-n-1}y^n\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle + n\left\langle x^{m-n}y^{n-1}\frac{\mathrm{d}y}{\mathrm{d}t}\right\rangle. \tag{16}$$

On substituting (9), (8) and (14) into (16), one obtains

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle x^{m-n}y^n\rangle &= (m-n)\langle x^{m-n-1}y^{n+1}\rangle - n\gamma(1-\gamma Dn)\langle x^{m-n}y^n\rangle - n\omega^2\langle x^{m-n+1}y^{n-1}\rangle \\
&\quad + na\sin(\Omega t)\langle x^{m-n}y^{n-1}\rangle + n(n-1)D_1\langle x^{m-n}y^{n-2}\rangle.
\end{aligned} \tag{17}$$

Equation (17) reduces to (15) for the special cases $n=0, m=2$, $n=2, m=2$ and $n=1, m=2$, as it should be.

The correlation functions can be found along the same lines as was done in (10)–(15) for the second moments by multiplying the first of equations (10) by $x(t_1)$, the second one (with $f=0$) by $y(t_1)$, and averaging the resulting equations, which gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t_1)x(t)\rangle &= \langle x(t_1)y(t)\rangle \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t_1)y(t)\rangle &= -\gamma\langle x(t_1)y(t)\rangle - \gamma\langle \xi(t)x(t_1)y(t)\rangle - \omega^2\langle x(t_1)x(t)\rangle + a\langle x(t_1)\rangle\sin(\Omega t).
\end{aligned} \tag{18}$$

The new correlators $\langle \xi(t)x(t_1)y(t)\rangle$ and $\langle \xi(t)x(t_1)x(t)\rangle$ can be found by using equation (7) leading to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi(t)x(t_1)x(t)\rangle &= \langle \xi(t)x(t_1)y(t)\rangle - \lambda\langle \xi(t)x(t_1)x(t)\rangle \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi(t)x(t_1)y(t)\rangle &= -\gamma\langle \xi(t)x(t_1)y(t)\rangle - \gamma\langle \sigma^2\rangle\langle x(t_1)y(t)\rangle \\
&\quad - \omega^2\langle \xi(t)x(t_1)x(t)\rangle - \lambda\langle \xi(t)x(t_1)y(t)\rangle + a\langle \xi(t)x(t_1)\rangle\sin(\Omega t).
\end{aligned} \tag{19}$$

In the following sections, the formulae obtained above will be applied to different special cases.

3. Force-free oscillator

3.1. White noise

The stationary ($\mathrm{d}/\mathrm{d}t \ldots = 0$) second moments of the force-free ($a = 0$) oscillator are obtained from equations (15), which gives

$$\langle xy\rangle_{\text{eq}} = 0 \qquad \langle y^2\rangle_{\text{eq}} = \omega^2\langle x^2\rangle \qquad \langle x^2\rangle = \frac{D_1}{\omega^2\gamma(1-2\gamma D)}. \tag{20}$$

For vanishing multiplicative noise, $D = 0$, equations (20) reduce to their standard form describing Brownian motion. In the presence of this noise, equation (20) shows that an (energetic) instability occurs when $2\gamma D > 1$. This result is different from that obtained for the case of the random frequency, equation (1), where the instability condition of the second moment has the form $\omega^2 D > \gamma$ [5, 18].

3.2. Dichotomous noise

For dichotomous noise there is a need to use the six equations (11)–(13), which we will consider in the equilibrium state ($\mathrm{d}/\mathrm{d}t \ldots = 0$) and in the absence of an external force ($a = 0$). Their solutions are

$$\langle x^2\rangle_{\text{eq}} = \frac{2D_1\{(\lambda+\gamma)[4\omega^2+\lambda(\lambda+2\gamma)]+2\lambda\gamma^2\sigma^2\}}{2\gamma(\lambda+\gamma)\omega^2[4\omega^2+\lambda(\lambda+2\gamma)]-4\omega^2\gamma^2\sigma^2[2\omega^2+\lambda(\lambda+2\gamma)]}. \tag{21}$$

In the limit case of white noise ($\sigma^2 \to \infty$, $\lambda \to \infty$ with $\sigma^2/\lambda = D$) equation (21) reduces to (20) as it should. For dichotomous noise, the second moment (21) is a non-monotonic function of both the noise strength σ^2 and its correlation rate λ.

4. Driven oscillator

All of the preceding was related to the field-free case which corresponds to equations (15) with $a = 0$. From now on, we are interested in the response of the underdamped harmonic oscillator to a periodic field, and, for simplicity, we present here only the case of white noise. The more cumbersome expression for dichotomous noise will not be given here.

4.1. Second moments

From equations (15) with $D_1 = 0$, one easily finds the third-order differential equation for $\langle x^2\rangle$,

$$\begin{aligned}\frac{\mathrm{d}^3\langle x^2\rangle}{\mathrm{d}t^3} &+ \gamma(3-5\gamma D)\frac{\mathrm{d}^2\langle x^2\rangle}{\mathrm{d}t^2} + 2[2\omega^2+\gamma^2(1-\gamma D)(1-2\gamma D)]\frac{\mathrm{d}\langle x^2\rangle}{\mathrm{d}t}\\ &+4\gamma(1-2\gamma D)\omega^2\langle x^2\rangle\\ &= 4a[\langle y\rangle + \gamma(1-2\gamma D)\langle x\rangle]\sin(\Omega t) + 2a\frac{\mathrm{d}}{\mathrm{d}t}[\langle y\rangle\sin(\Omega t)].\end{aligned} \tag{22}$$

The first moments, $\langle x\rangle$ and $\langle y\rangle = \mathrm{d}\langle x\rangle/\mathrm{d}t$, that enter equation (22) consist of two parts, $\langle x\rangle_0$ and $\langle x\rangle_a$, where the first one is the solution of the field-free equation (5) and the second one is defined by (6). In the stable region ($\gamma D < 1$), the solution of equation (5) vanishes as $t \to \infty$, and the stationary solution of equation (22) ($\mathrm{d}/\mathrm{d}t \cdots = 0$) is obtained on substituting $\langle x\rangle = \langle x\rangle_a = A\sin(\Omega t + \phi)$ and $\langle y\rangle = \mathrm{d}\langle x\rangle/\mathrm{d}t = A\Omega\cos(\Omega t+\phi)$. Thus, one gets for the stationary value of the second moment,

$$\langle x^2\rangle_{\text{st}} = \frac{aA}{2\omega^2}\left(\cos\phi - \frac{\Omega\sin\phi}{\gamma(1-2\gamma D)}\right) \tag{23}$$

where A and $\tan\phi$ are defined in (6). It follows from (23) that the second moment, $\langle x^2\rangle_{st}$, is a non-monotonic function of the noise strength D. It turns out that for dichotomous noise the second moment shows a non-monotonic dependence on both the noise strength σ^2 and the correlation rate λ.

4.2. Correlation functions

From equations (18), (19), one can find the fourth-order differential equation for $z \equiv \langle x(t_1)x(t)\rangle$,

$$\frac{d^4z}{dt^4} + 2(\lambda+\gamma)\frac{d^3z}{dt^3} + [2\omega^2 + \gamma^2 + 3\lambda\gamma + \lambda^2 - \gamma^2\langle\xi^2\rangle]\frac{d^2z}{dt^2} + [(2\omega^2 + \lambda\gamma)(\lambda+\gamma) - \lambda\gamma^2\langle\sigma^2\rangle]\frac{dz}{dt} + \omega^2[\omega^2 + \lambda(\lambda+\gamma)]z = a\left[\frac{d^2}{dt^2} + (2\lambda+\gamma)\frac{d}{dt} + (\omega^2 + \lambda^2 + \lambda\gamma)\right]\langle x(t_1)\rangle \sin(\Omega t) - a\gamma\left(\frac{d}{dt} + \lambda\right)\langle\xi(t)x(t_1)\rangle\sin(\Omega t). \quad (24)$$

In order to avoid the need for the rather long formulae, we restrict our attention to the case of white noise. With equations (8), (9), one gets from (19)

$$\langle\xi(t)x(t_1)x(t)\rangle = 0 \qquad \langle\xi(t)x(t_1)y(t)\rangle = -\gamma D\langle x(t_1)y(t)\rangle \quad (25)$$

and the substitution of these formulae into (18) leads to

$$\left[\frac{d^2}{dt^2} + \gamma(1-\gamma D)\frac{d}{dt} + \omega^2\right]\langle x(t_1)x(t)\rangle = Aa\sin(\Omega t_1 + \phi)\sin(\Omega t). \quad (26)$$

The form of a solution of equation (26) depends on the type of the initial conditions which can be $\langle x(t_1)x(t)\rangle$ either at $t = t_1$ or at $t = 0$. In deciding on the latter, one assumes that the initial condition $x\ (t = 0)$ is not random. Then, $\langle x(t_1)x(t=0)\rangle = x(t=0)\langle x(t_1)\rangle$ and $(d/dt)\langle x(t_1)x(t=0)\rangle = (dx/dt)(t=0)\langle x(t_1)\rangle$. Solving the inhomogeneous equation (26) through the use of the Green function, one obtains

$$\langle x(t_1)x(t)\rangle = \exp\left(-\frac{\gamma(1-\gamma D)t}{2}\right)\left\{x(t=0)\langle x(t_1)\rangle\left[\cos(\omega_1 t) + \frac{\gamma(1-\gamma D)}{2\omega_1}\sin(\omega_1 t)\right] + \frac{1}{\omega_1}\frac{dx}{dt}(t=0)\langle x(t_1)\rangle\sin(\omega_1 t)\right\} + Aa\sin(\Omega t_1 + \phi) \times \int_0^t \exp\left[-\frac{\gamma(1-\gamma D)}{2}(t-\theta)\right]\cos[\omega_1(t-\theta)]\sin(\Omega\theta)\,d\theta \quad (27)$$

where $\omega_1 = \sqrt{\omega^2 - (\gamma^2(1-\gamma D)^2)/4}$, and the last integral in (27) can be easily expressed in terms of elementary functions.

For white noise the correlation function (27), like the second moment $\langle x^2\rangle$, is a non-monotonic function of the noise strength D. For dichotomous noise the correlation function shows a non-monotonic dependence on both the noise strength σ^2 and the correlation rate λ.

5. Periodically varying damping

It is instructive to compare the influence of multiplicative noise with the effects associated with an oscillator with a periodically changed damping parameter described by the following equation:

$$\frac{d^2x}{dt^2} + \gamma[1 + b\sin(\Omega_1 t)]\frac{dx}{dt} + \omega^2 x = 0. \quad (28)$$

Equation (28) with periodic coefficients has the Floquet solution in the form [19]

$$x(t) = \exp(\alpha t)\psi(t) = \exp(\alpha t)\sum_{n=0}^{\infty} A_n \sin\left(\frac{n\Omega_1 t}{2}\right) + B_n \cos\left(\frac{n\Omega_1 t}{2}\right) \tag{29}$$

where the periodic function $\psi(t)$ is expanded in a Fourier series. As is evident from (29), $\psi(t)$ vanishes at $x \to \infty$ for $\alpha < 0$, diverges for $\alpha > 0$, and remains a bounded periodic function for $\alpha = 0$. Hence, $\alpha = 0$ defines the stability boundary of the stationary solutions of equation (28). On the substitution of (29) with $\alpha = 0$ into (28), and comparing the harmonics in front of the sine and cosine terms, one obtains the infinite systems of linear equations for A_n and B_n, which have nonzero solutions if the infinite determinant of these equations $\Delta(\alpha = 0)$ vanishes, $\Delta(\alpha = 0) = 0$. One has to truncate this determinant at some n, and afterwards to improve the result by taking into account larger values of n. Leaving only terms with $n = 1$, one obtains the following equations:

$$\begin{aligned} &\left(\omega^2 - \frac{\Omega_1^2}{4} + \frac{\Omega_1 \gamma b}{4}\right) A_1 + \frac{\Omega_1 \gamma}{2} B_1 = 0 \\ &-\frac{\Omega_1 \gamma}{2} A_1 + \left(\omega^2 - \frac{\Omega_1^2}{4} - \frac{\Omega_1 \gamma b}{4}\right) B_1 = 0. \end{aligned} \tag{30}$$

Equations (30) have nontrivial solutions if the determinant of these equations vanishes, which gives

$$b = \sqrt{4 + \frac{\left(4\omega^2 - \Omega_1^2\right)^2}{\gamma^2 \Omega_1^2}}. \tag{31}$$

The stability boundary (31) of the solution $x = 0$ has a V form with a stable state located inside this curve. Equation (31) defines a necessary condition for the nonzero periodic solution of the periodically varying velocity in the same way as the condition $\gamma D = 1$ defines it for a random velocity. The difference is that in the former case an external field defines the basic frequency Ω_1 of oscillation, while in the latter case the oscillations occur at the oscillator frequency ω.

6. Conclusions

Multiplicative noise can have a twofold impact on a harmonic oscillator. The case of a random frequency has received the most study, while a random damping parameter has been discussed only recently in connection with velocity fluctuations in convective media. The distinction between these two cases makes itself evident already in the behaviour of the average coordinate ('first moments') subjected to white noise. This moment remains unchanged in the former case, and diverges as $t \to \infty$ showing instability in the case of a random damping parameter. Note that for colour noise the first moment may diverge for the case of a random frequency as well.

Equations of motion for the averaged squared coordinate ('second moments') and the correlation functions have been solved for zero initial conditions, while in the absence of an external field the stationary motion is defined by additive noise. An approximate procedure has been used for the splitting of high-order correlators, which allows one to obtain a closed system of differential equations for white and dichotomous noise. The second moments and the correlation functions for colour (dichotomous) noise are non-monotonic functions of the noise strength and correlation rate, showing the stochastic resonance phenomenon. Periodically varying damping (parametric excitation) can also result in an instability with the basic frequency of excitation, which is different from the oscillations with the oscillator frequency induced by randomly varying damping.

Further theoretical work will include calculations for different types of colour noise, the Lyapunov indices, as well as some other problems considered for the case of random frequency [4].

As was already indicated, our calculations are specific to the experimental situations described in [8–13]. For example, the motion of vortices in superconducting films is described [13] by the Ginzburg–Landau equation which is nothing but our equation (2) with the coordinate x and time t replaced by the order parameter and coordinate, respectively. The term with damping in equation (2) corresponds to the bias current applied to a superconducting film. Then, the different moments of the order parameter can be found by magneto-optical measurements of high temporal resolution [20].

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